Thinking

<u>Math</u>ematically

Why think Mathematically?

- **Be able to successfully describe problems**
  - Defining the terms you use, requiring others to do the same.
  - Agreeing on what a solution looks like
  - Verify whether a statement is really true
  - Having confidence that you are right
- **Develop effective thinking habits**
  - Recognizing general patterns
  - Being able to apply the right tools for the job
  - Breaking down problems into steps.
  - Assuming hypotheticals.

A whirlwind history of ***numbers***

and how they are useful

Sets

- **In the beginning, there was counting. And it was good.**
  - Set notation $\boldsymbol{A}$ ={  ,  ,  } $\boldsymbol{B}$ = {  ,  ,  }
  - Be able to communicate a rule which determines set membership.
  - Variables can stand for any member of the set: Let $x \in \boldsymbol{A}$, $y \in \boldsymbol{B}$
  - The above sets both have three objects. They share that property.
- **Sets of Numbers**
  - $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ Natural numbers
  - $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ Integers
  - $\mathbf{Q} = \{ p/q : p \in \mathbf{Z}, q \in \mathbf{N}, q \neq 0 \}$ Rational numbers
  - **R** – real numbers, **C** – complex numbers

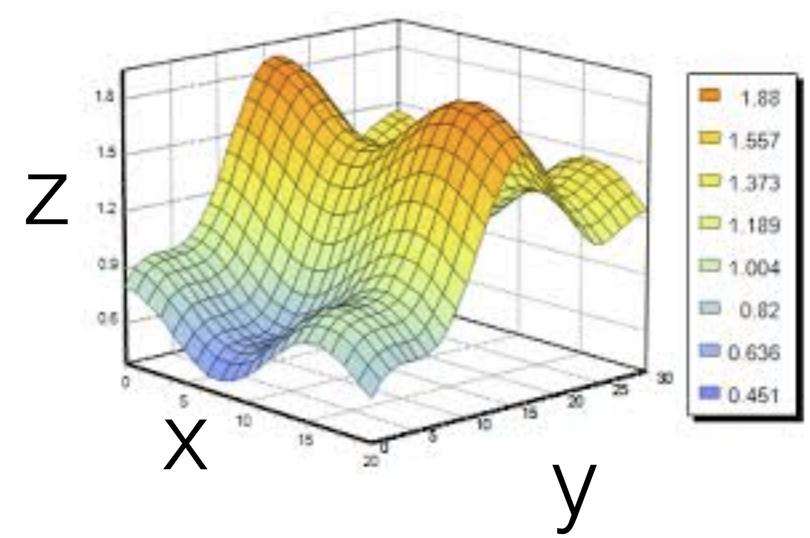
Functions

- **Cross Product**
  - ***A*** $\times$ ***B*** means: the set of all possible tuples $\{ (a, b) : a \in \boldsymbol{A}, b \in \boldsymbol{B} \}$
  - ***A*** $\times$ ***B*** $\times$ ***C*** is the set of all possible triples $\{ (a, b, c) : a \in \boldsymbol{A}, b \in \boldsymbol{B}, c \in \boldsymbol{C} \}$
- **"Function"** is a concept to represent operations on variables:

output

$$f(\,x, y\,) = x + 3y$$

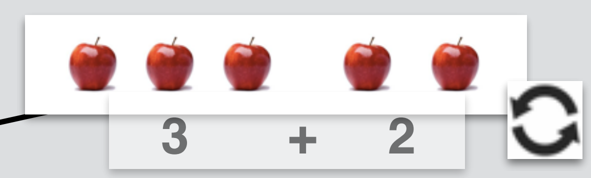
inputs (aka parameters)



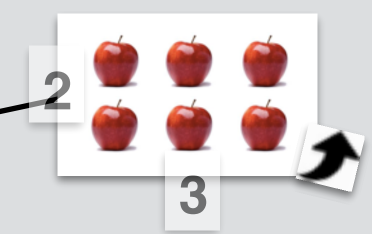
- As you vary the input, the output varies
  - Let's say $x \in \mathbf{R}$, $y \in \mathbf{R}$, and the output is a real number too
  - Then we can write $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, and draw the graph of f in 3 dimensions

Algebra on N

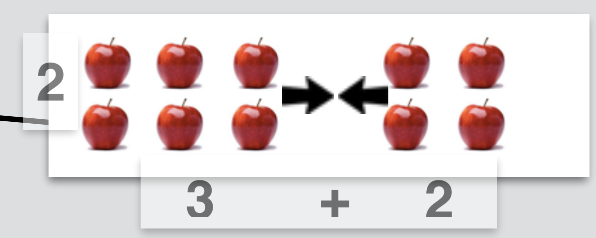
- Remember, each variable has to come from some set
  - "Let ***a, b*** be any natural numbers" notation: Let $a \in \mathbf{N}, b \in \mathbf{N}$
- Addition on Natural Numbers
  - It is commutative $\quad a + b = b + a$



- Multiplication on Natural Numbers
  - It is commutative $\quad a \bullet b = b \bullet a$



  - Distributive law $\quad a \bullet (x + y) = a \bullet x + a \bullet y$



Addition and Multiplication are functions of two variables

$f(a, b) = a + b$

$g(a, b) = a \bullet b$

Algebra on N

- Raising number to a power $\boldsymbol{n} \in \boldsymbol{N}$ is repeated multiplication, just like multiplication is repeated addition:

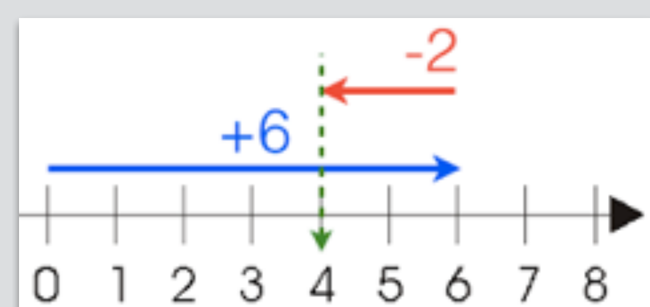
$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a \cdot a}_{n\ times}$$

- So we have similar results as with multiplication:
  - $a^{m+n} = a^m a^n$
  - $(a^m)^n = (a^n)^m$
  - $(a^m)^n = a^{(mn)}$

$$(a^m)^n = \begin{array}{c|l} & \overbrace{\qquad\qquad\qquad\qquad}^{m\ times} \\ n\ times & (a \cdot a \cdot a \cdot \ldots \cdot a \cdot a) \cdot \\ & (a \cdot a \cdot a \cdot \ldots \cdot a \cdot a) \cdot \\ & \quad\vdots \\ & (a \cdot a \cdot a \cdot \ldots \cdot a \cdot a) \end{array}$$

From Natural Numbers to Integers

- Find a that satisfies $a+5=3$.
- **N** won't work, we need new types of numbers: **Z** (integers).
- It turns out we can do algebra with these numbers.
- Since $a+0=a$ for any number a, we call 0 the **additive identity**
- The "negative" of a will be denoted $-a$.
  It is the unique number which makes $a+(-a)=0$
- And we define "subtraction" to be "adding the negative": $a-b=a+(-b)$
- Thus we extend algebra to the integers.
- By the way, $|a|$ means "absolute value of a"

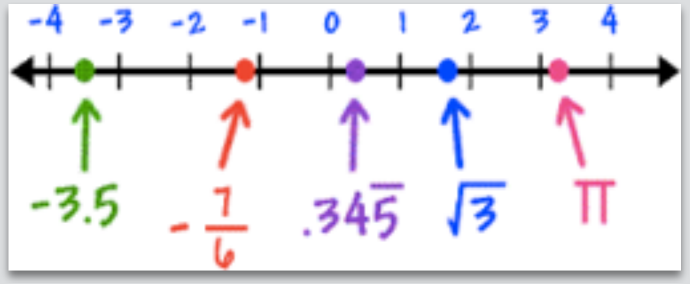


Algebra on Q

From Integers to Rational Numbers

- Find a that satisfies $a \bullet 5 = 3$.
- **Z** won't work, we need new types of numbers: **Q** (rational numbers).
- It turns out we can do algebra with these numbers.
- Since $a \bullet 1 = a$ for any number a, we call 1 the **multiplicative identity**
- The "reciprocal" of a will be denoted a^{-1} which is same as $1/a$
  It is the unique number which makes $a \bullet a^{-1} = 1$
- And we define "division" to be "multiplying by the reciprocal":

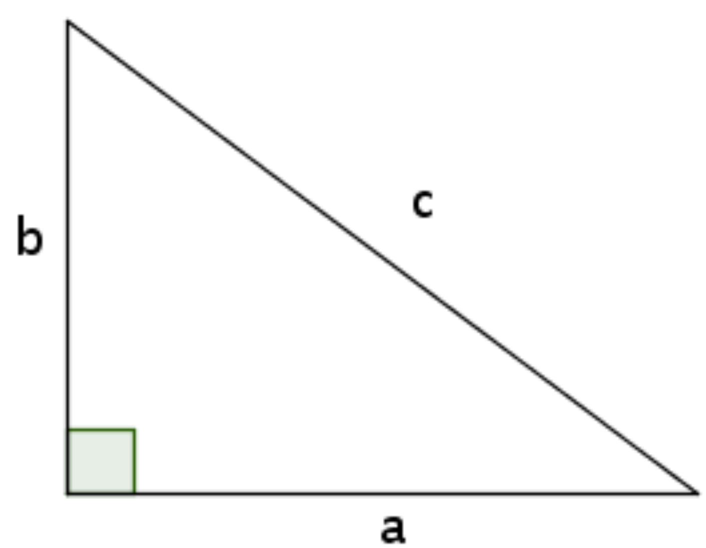
  $a / b = a \bullet b^{-1} = a \bullet 1/b$

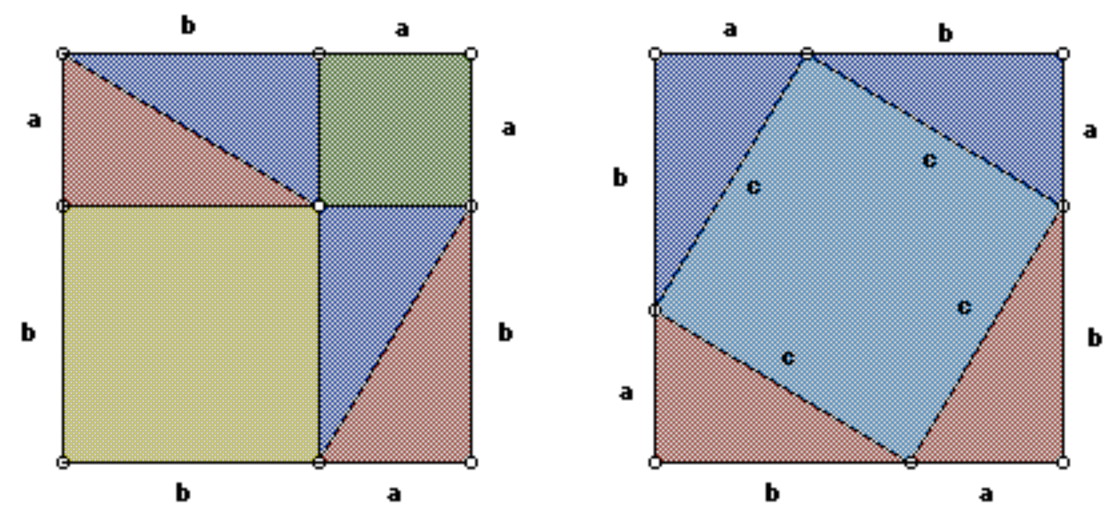


- Thus we extend algebra to the rational numbers.

Geometry

Pythagorean theorem

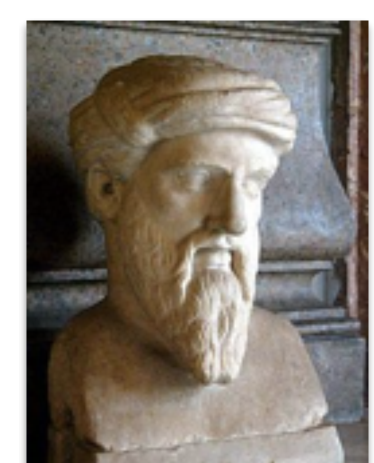




Proof

In a right triangle,
with sides *a* and *b*
and hypotenuse *c*,

$$a^2 + b^2 = c^2$$



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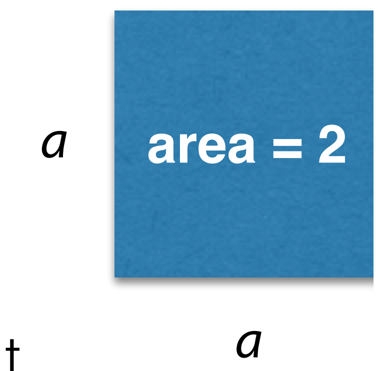
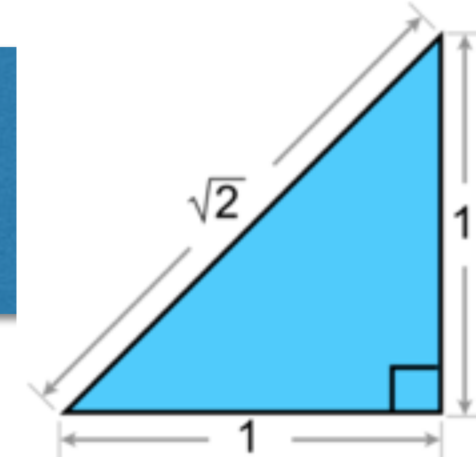
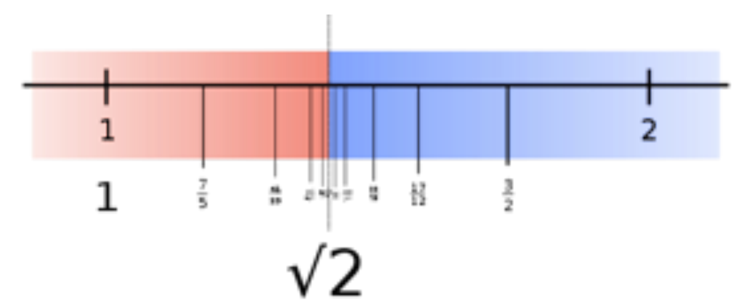
Irrational Numbers

From Rational Numbers to Real Numbers





- Find a that satisfies $a^2 = 2$.
- It turns out that no rational number $a = p / q$ can satisfy it.
- The ancient Greeks knew 2,500 years ago!
- When we discuss number theory, I will show the proof.
- The inverse operation of raising to a power is raising to a fractional power.



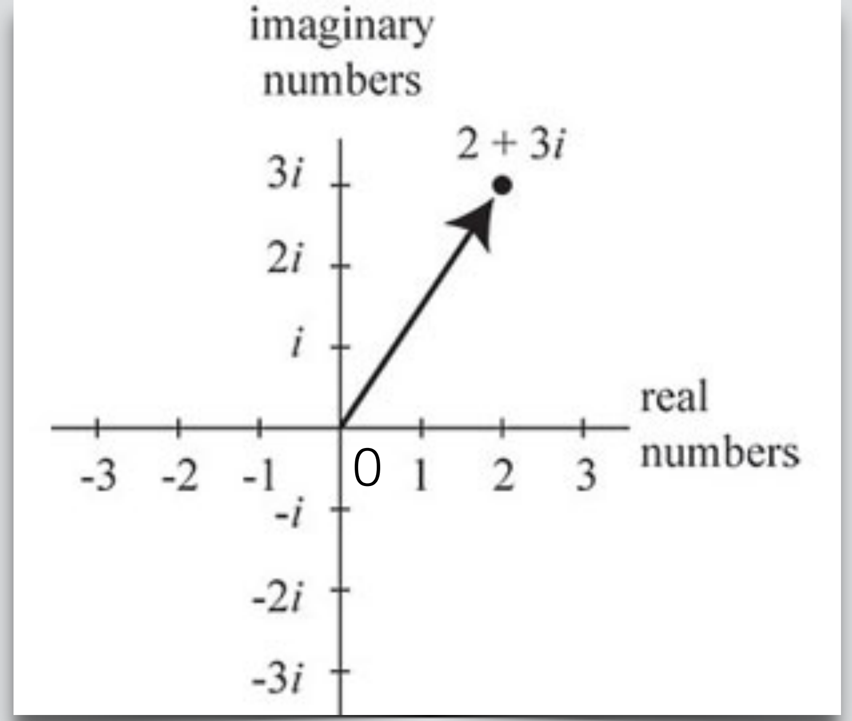
- Do you remember we had $\boldsymbol{(a^m)^n = a^{(mn)}}$?
- $(\sqrt{a}\,)^2 = a$ or put another way $(a^{1/n})^n = a$

In a future lesson, we will actually explore the set of Real Numbers, R.
For now, it is enough to know that we can do algebra with them like Q.

Algebra on C

<u>From Real to Complex Numbers</u>

- Find a that satisfies $a^2 = -1$
- **R** won't work, we need new types of numbers: **C** (complex numbers).
- It turns out we can do algebra with these numbers! All the usual rules work with **C**.
- Define $\underline{i} = \sqrt{-1}$. On the right, you can see the "imaginary axis" of real numbers multiplied by i .
  All the usual rules still work!



real part

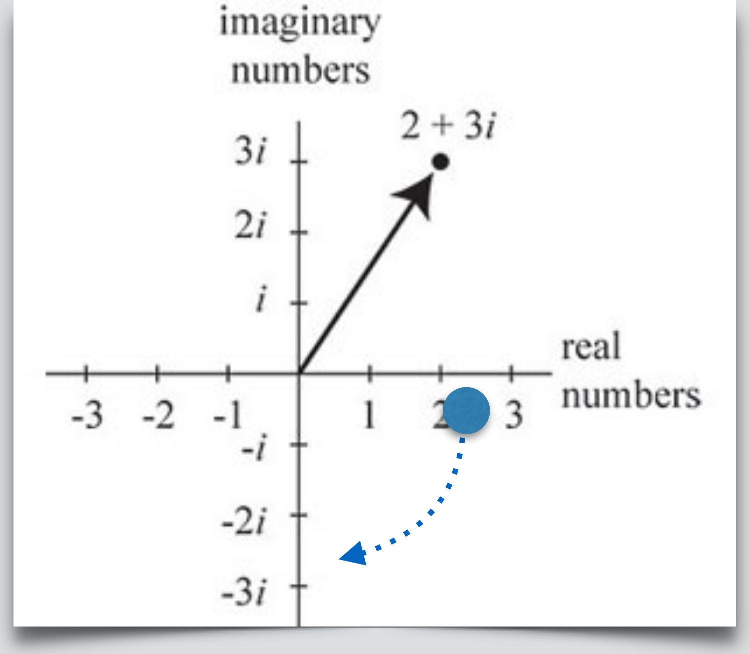
imaginary part

- Addition is $(a + \underline{bi}) + (c + \underline{di}) = (a + c) + \underline{(b + d)\,i}$
- Multiplication $(a + \underline{bi})\,(c + \underline{di}) = (ac - bd)\,\underline{(ad + bc)\,i}$
- 1 is still the **multiplicative identity**, because $a \cdot 1 = a$ for all $a \in$ **C**
- **0** is still the **additive identity**, because $a + 0 = a$ for all $a \in$ **C**
- The commutative and distribute laws still apply.

Graphing Algebra on C

Here is how it looks visually

- Adding real numbers together is like adding two vectors together.
- Adding a real number moves the point horizontally.
  Adding an imaginary number moves the point vertically.
- Multiplying by *i* rotates 90% counterclockwise
  Multiplying by –*i* rotates 90% clockwise.

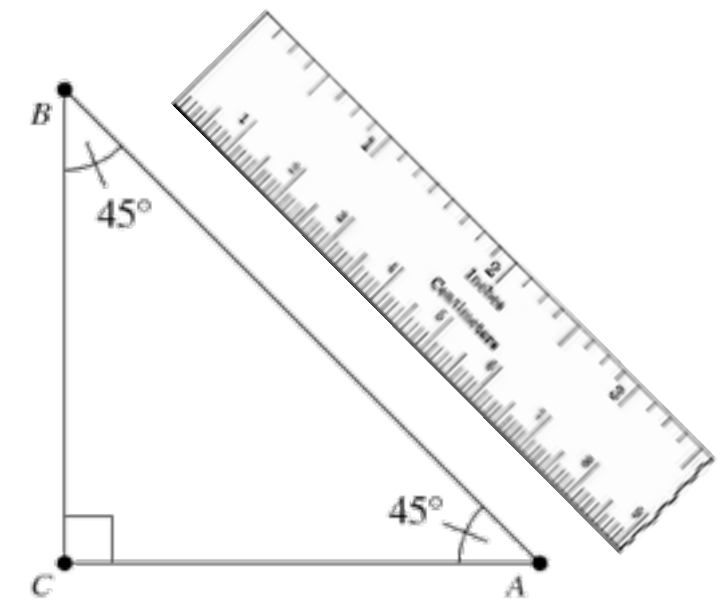


- Why? Because by convention,
  +*i* is up and –*i* is down.
- You may wonder, what is $\sqrt{i}$?
  It is the number you get by rotating 1 by 45 degrees counterclockwise.

Existence

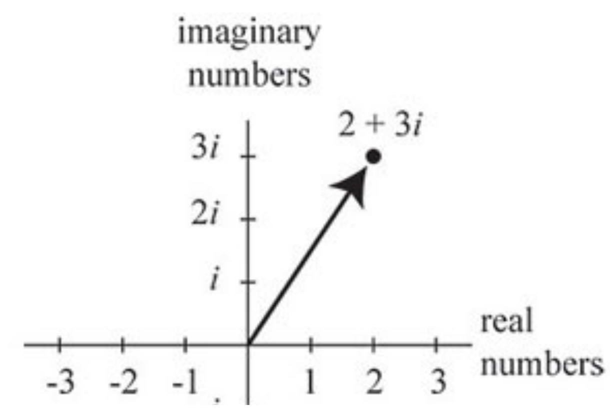
Do these numbers really exist?

That is an interesting question. The ancient Greeks were very disturbed that no matter how much you subdivided a ruler into equal parts, you could never have it measure both the sides and the hypotenuse of a simple isosceles right triangle.

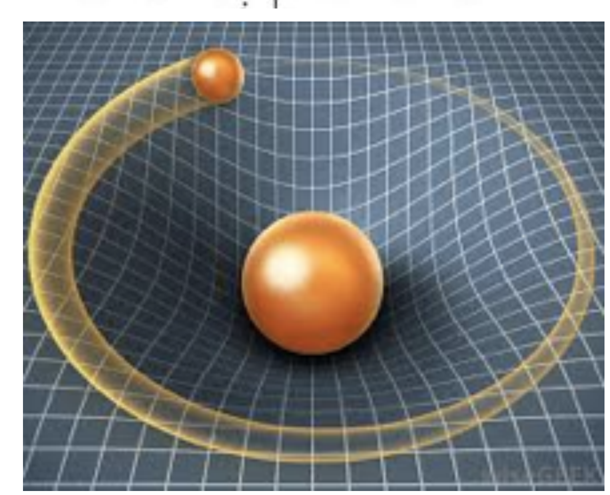


One can imagine people might have had trouble accepting negative numbers exist.

The imaginary numbers are called imaginary because people were already comfortable with the real number line when they were invented. So they named the other the "imaginary axis".



Whether mathematical ideas like gravity fields really exist depends on what you mean by "existence".



What we can say is that they are **useful** for doing mathematical calculations. The models we make to understand the world around us can be used to predict certain phenomena and explain how they behave.

Usefulness

N is used for counting, adding things, etc.

Z is used for debts, measuring units, etc.

Q is used for ratios, subdividing measuring units, etc.

R is used for when you want a smooth number line

C is when you want to solve any polynomial equation for *x*

Concept: Generalization

In this topic, we explored the motivations behind why people came to use more and more complex types of numbers. This illustrates one of the fundamental ways of thinking mathematically: trying to generalize a specific situation.

Even the counting numbers themselves are generalizations representing the idea that the ***types*** of objects don't have to matter, if all you care about is the size of the set. You can perform the same operations with a set of two apples as you can with a set of two oranges, so you can ***abstract away*** the type of objects in the set.

Similarly, variables represent the idea that the particular ***numbers*** in a number set don't matter, if all you care about is doing some operation with them. You just have to know the rules of algebra on that set.

And as we ran across different algebraic equations and needed to solve for x, we constructed ***larger*** sets of numbers that included all our previous numbers, and then some. We ***generalized*** the algebraic operations we already used (addition, multiplication, powers etc.) so that they worked on the larger number system in a seamless, compatible way with the original number system.

All these are examples of how mathematicians go about ***generalizing*** about things.