Thinking

***Math**ematically*

Sets

- Sets are collections of things, with the following properties:
  - ***Order doesn't matter***: $\{1, 2, 3\} = \{1, 3, 2\}$
  - ***Repetition doesn't matter:*** $\{1, 2, 3\} = \{1, 2, 2, 3, 3, 3\}$
  - You can define sets by using variables from other sets:
    - $\{\, 2n + 1 : \quad n \in \mathbf{N} \,\}$ Positive odd numbers
    - $\{\, p / q \ : \quad p \in \mathbf{Z}, q \in \mathbf{N}, q \neq 0 \,\}$ All rational numbers
- Two sets are equal when they have the exact same elements.

Subsets

- We say that $A \subseteq B$, "*A* is a subset of *B*" if $p \in B$ whenever $p \in$ A
  - {1, 2, 3} is a subset of {1, 2, 3, 4, 5} but not vice versa
  - When $A \subseteq B$ and $B \subseteq A$ then $A = B$
- The empty set { } is the set with no elements
  - Given any set *A*, { } is a subset of it

Numbers

- **Recall our sets of numbers:**
  - $\mathbf{N} = \{0, 1, 2, 3, \dots\}$ Natural numbers
  - $\mathbf{Z} = \{ \dots, -2, -1, 0, 1, 2, \dots \}$ Integers
  - $\mathbf{Q} = \{ p / q \ : p \in \mathbf{Z}, q \in \mathbf{N}, q \neq 0 \}$ Rational numbers
  - $\mathbf{R}$ = Rational + Irrational numbers Real numbers
  - $\mathbf{C} = \{ a+ib \ : a \in \mathbf{R}, b \in \mathbf{R} \}$ Complex numbers
- So $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R} \subseteq \mathbf{C}$

Functions

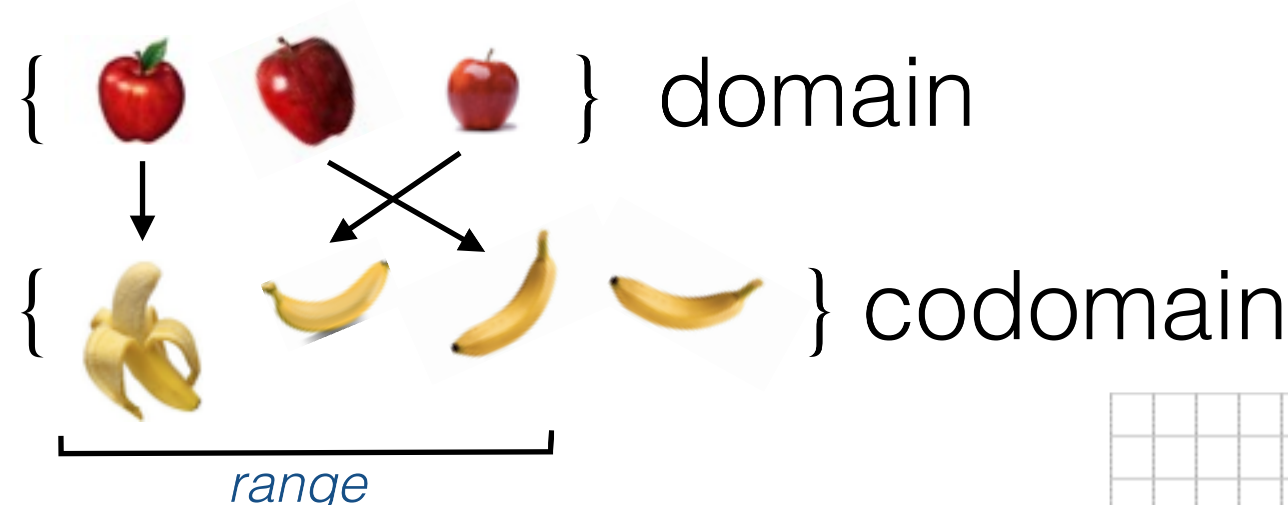
- A **"Function"** is an abstract concept that we can use to represent any rule, machine, however complex, that takes an input, and gives an output.

- A function $f: A \to B$ "maps" elements of the set A to elements of the set B.

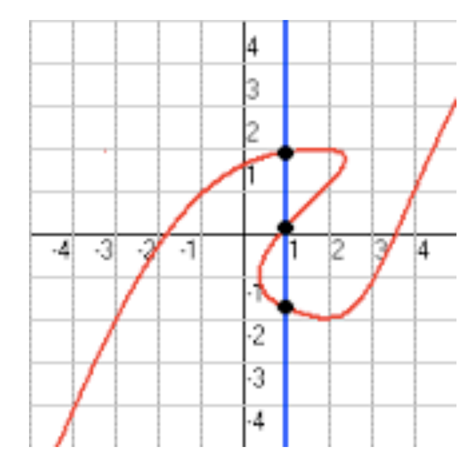
- As you vary the input, the output varies. Given variables $a \in A$, $b \in B$ it acts like this: $a \xrightarrow{f} b$

Functions

- A function "maps" members of its ***domain*** to its ***codomain***
  - Let A be a set of 3 Apples, and B be a set of 4 Bananas
  - $f : A \rightarrow B$

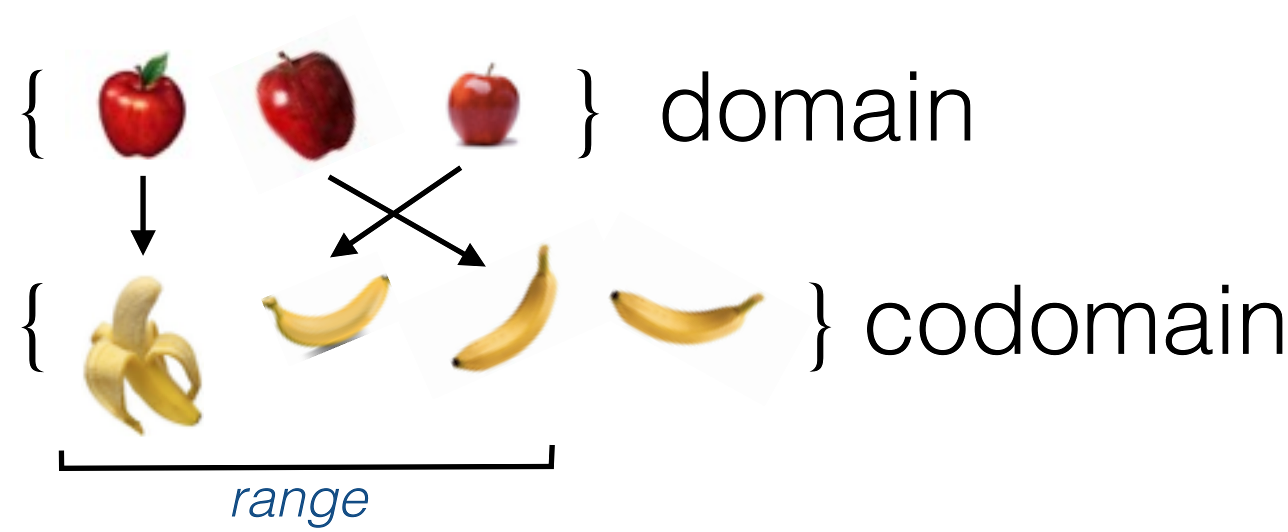


- The set of all outputs of f is called is ***range***
- Each input to f can have at most one output.
  For graphs of f: $\mathbf{R} \rightarrow \mathbf{R}$ we call this the vertical line test:



Properties of Functions

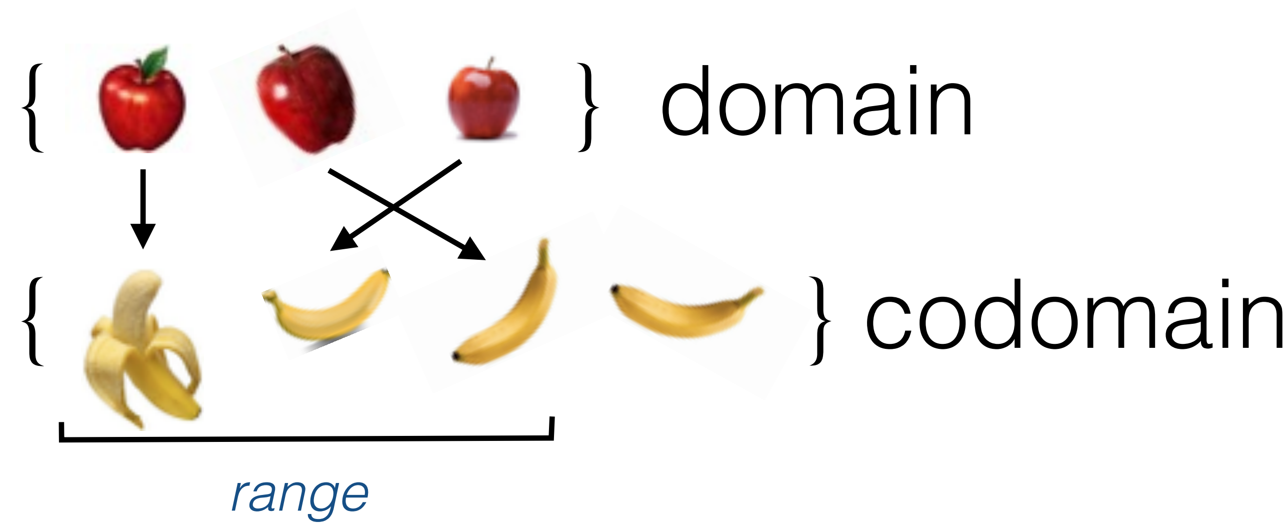
- When a function *f* is called ***onto*** it means:
  - ***its range = its codomain.*** Or in other words:
  - ***every output has <u>at least one</u> input that produces it***



The above function is <u>not</u> onto because one of the bananas is not in the range of *f*.

Properties of Functions

- When a function *f* is called ***one-to-one*** it means:
  - ***every output has <u>at most one</u> input that produces it***



The above function <u>is</u> one-to-one: every banana in the range of *f* has exactly one input that produces it.

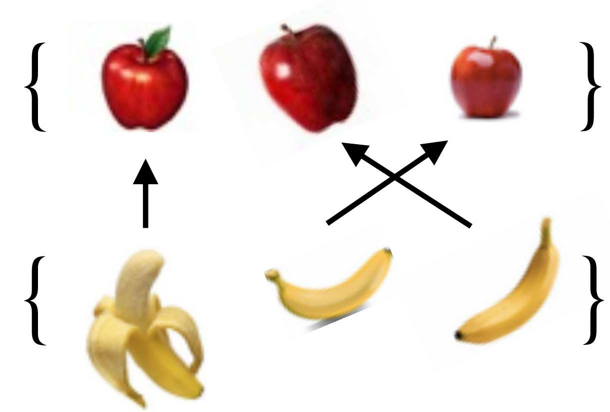
Invertible Functions

- When a function f is both ***one-to-one*** and ***onto***, then every output has **<u>exactly one input</u>** that produces it.

- Then we can define an ***inverse function*** f^{-1} which maps all the outputs back to their inputs, like so:

  - $f : A \rightarrow B$

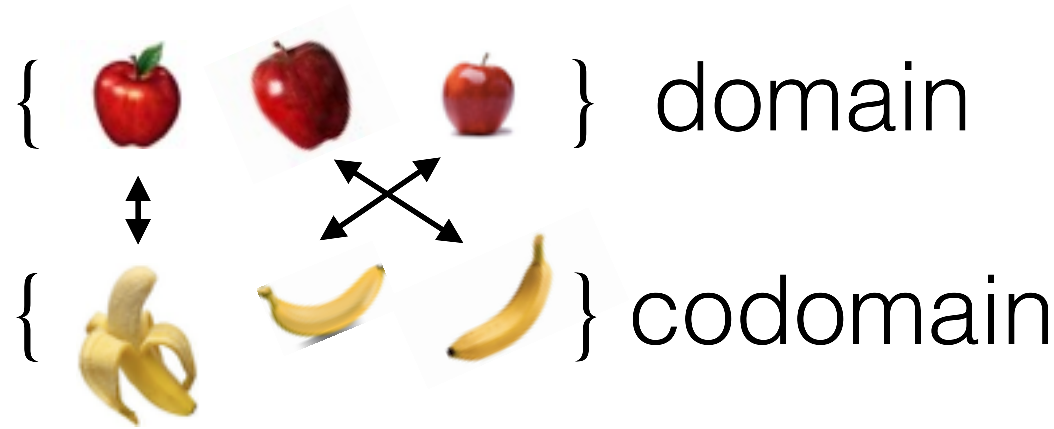
  - $f^{-1} : B \rightarrow \mathrm{A}$



domain

codomain

Invertible Functions



$$a \xrightarrow{f} b \xrightarrow{f^{-1}} a$$

For every $a \in A$, notice that $a = f^{-1}(\, f(a)\,)$

That means $f^{-1} \circ f = \mathrm{I}$, the identity function $\mathrm{I}(a) = a$

Similar to how $x^{-1} \bullet x = (1/x) \bullet x = 1$ for real numbers.

Sometimes people use the following names instead:

onto → ***surjective***

one-to-one → ***injective***

invertible → ***bijective***

The Size of a Set

For many sets, we can tell how many elements they contain by simply counting them.

If a set A contains m elements, we say $|A| = m$

If a set B contains $n > m$ elements, we know $|B| > |A|$

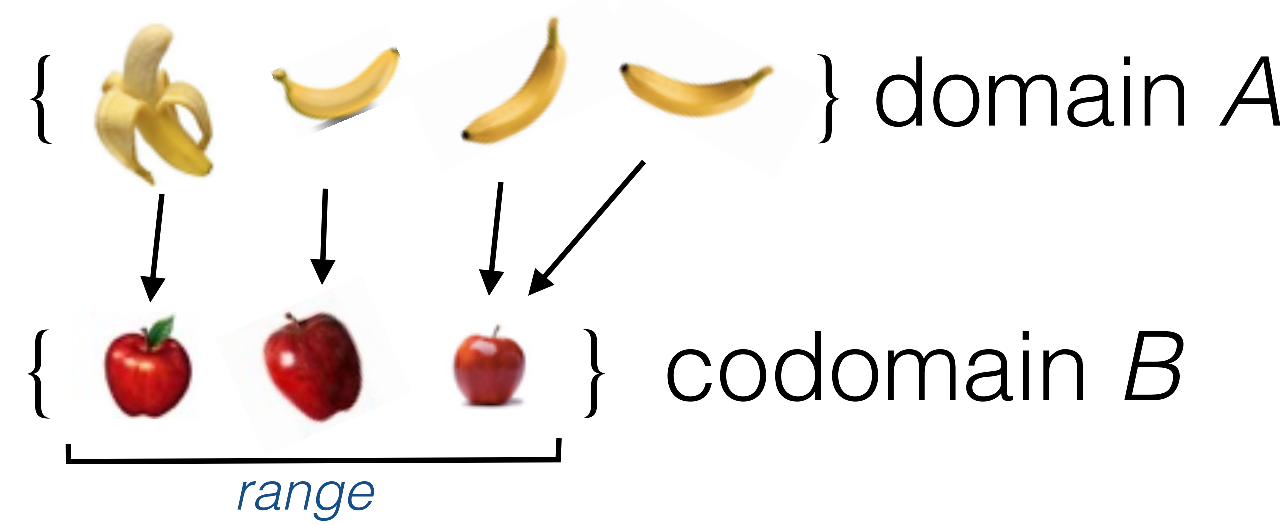
A ***finite*** set contains n elements, where $n \in \mathbf{N}$

But mathematicians often deal with ***infinite*** sets.
We can generalize the concept of ***size*** to those.

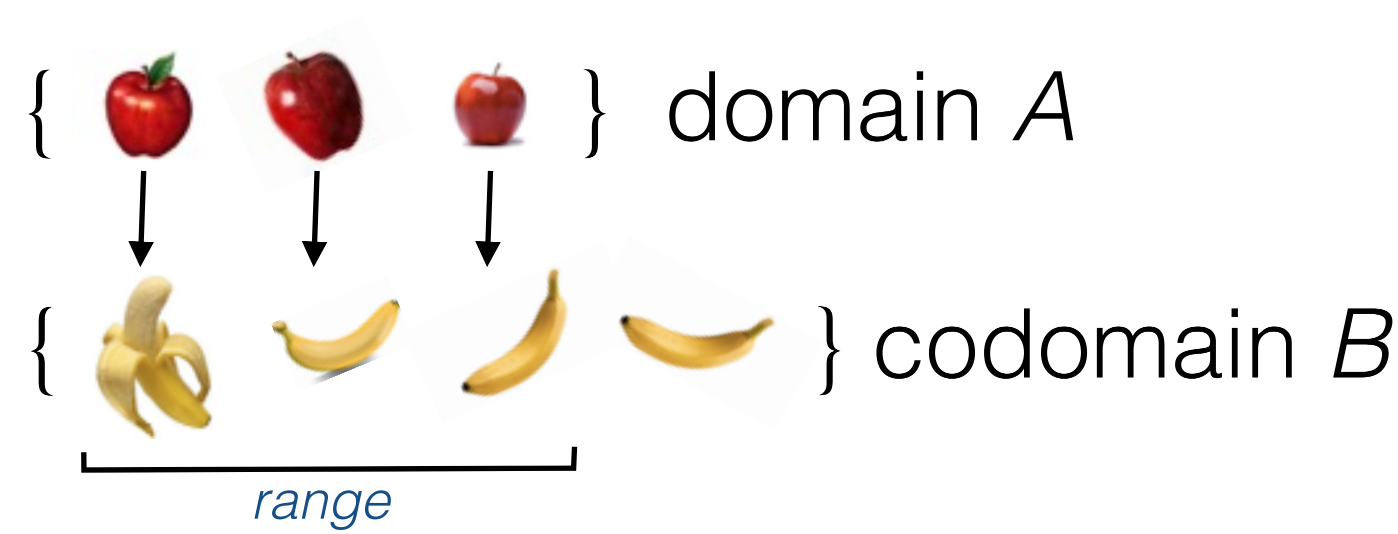
Cardinality

There are two ways to ***compare*** the sizes of sets. First way:

If at least one function *f* maps *A* ***onto*** *B*, then we say $|A| \geq |B|$, *i.e.* *A*'s size, or ***cardinality*** is at least that of *B*.



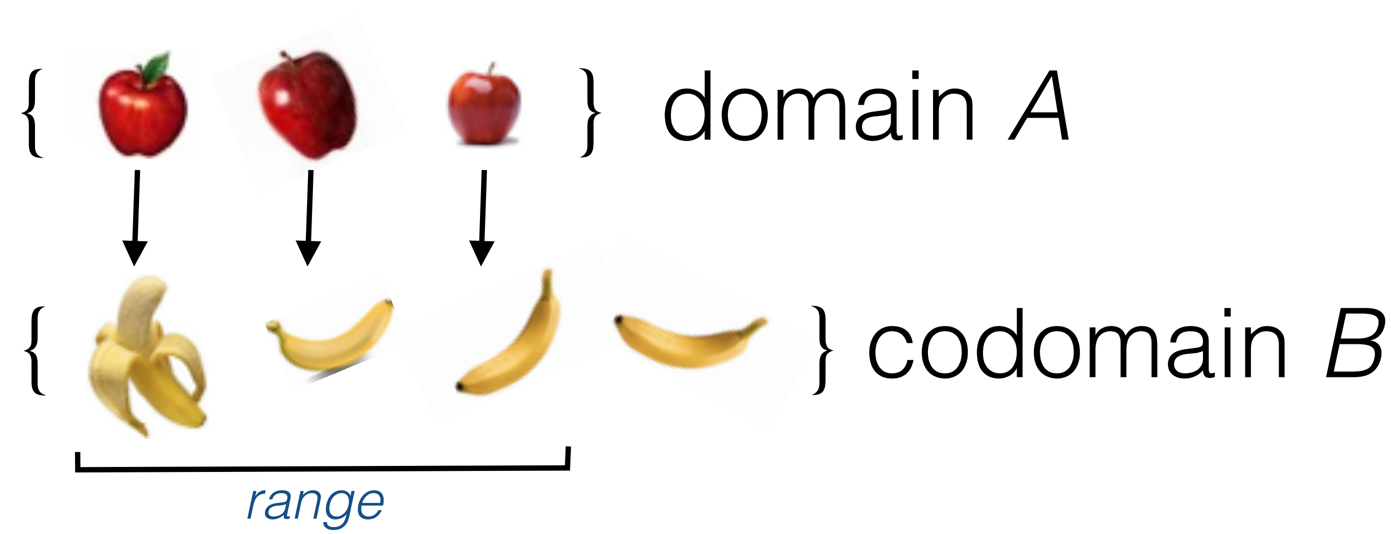
There is no function that maps *A* ***onto*** *B* here since each input has at most 1 output:



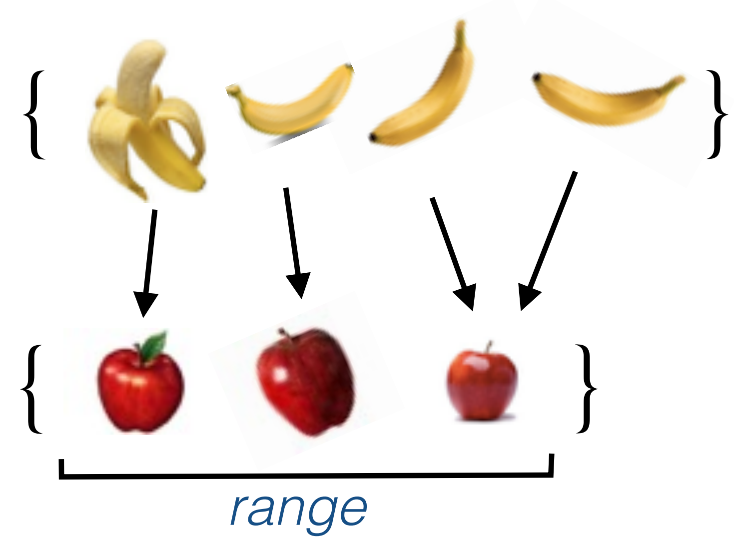
Cardinality

The first way used ***onto*** functions. Here is the second way:

If there is a ***one-to-one*** function $f: A \to B$, then we say $|A| \leq |B|$, *i.e.* A's size, or ***cardinality*** is at most that of B.



There is no ***one-to-one*** $f: A \to B$ since there are more inputs than outputs in this case:



Subsets and Cardinality

If $A \subseteq B$ then $|A| \leq |B|$, since we can easily find a one-to-one function $f : A \rightarrow B$, namely the identity function $\mathrm{I}(a) = a$.

So if we ***know*** $A \subseteq B$, and we want to show that $|A| = |B|$ we just need to show that $|A| \geq |B|$.

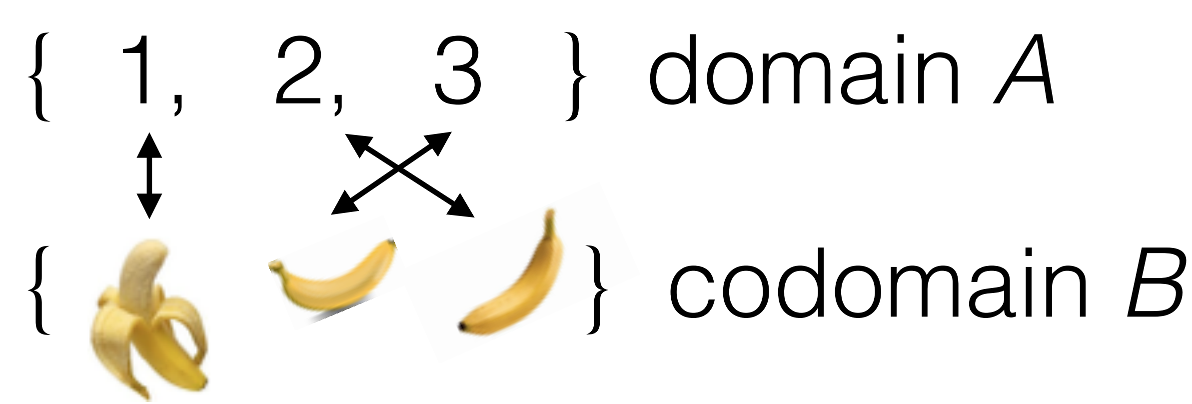
which we can do by finding a function f that maps A ***onto*** B.

Counting

If we can find an ***invertible*** function $f : A \to B$,
we know it's both one-to-one and onto,
so $|A| \leq |B|$ and also $|A| \geq |B|$ and thus $|A| = |B|$

In the case of finite sets, this is the same as ***counting:***

The act of counting these bananas involves making an invertible function from the set $A = \{1, 2, 3\}$ to the set B of these bananas.

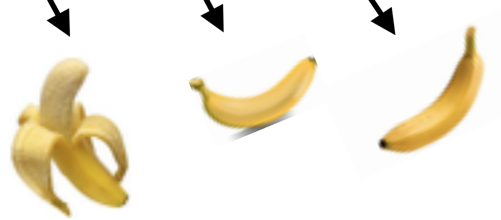


Infinite Sets

Some sets have an interesting property of being ***infinite.***

They cannot be counted using any natural number.

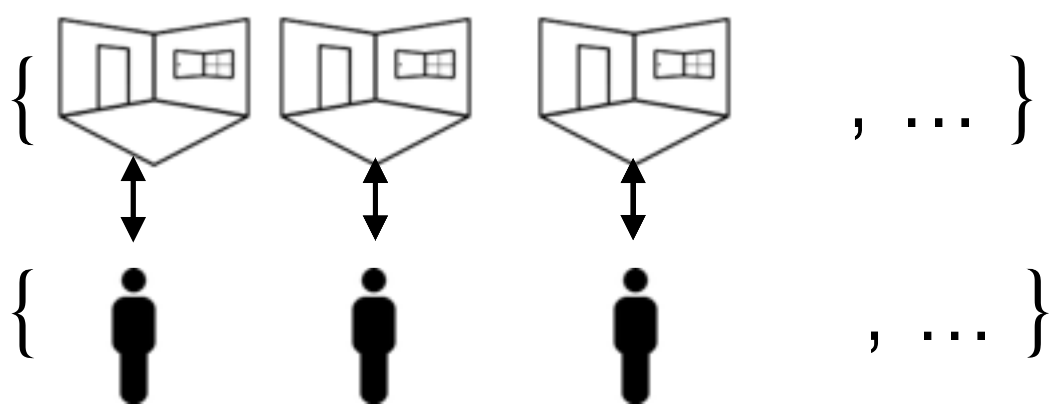
$\{1, \quad 2, \quad 3 \quad\}$ = domain A where $|A| = 3$

$\{$  $, \ldots\}$ codomain B *is infinite*

No matter what domain *A* you choose,
If the size of $|A|$ is some $n \in \mathbf{N}$, then it's too small:
there is no way to map *A* ***onto*** all of B.

Hilbert's Grand Hotel

Imagine a hotel with an infinite number of rooms.
Each room is already occupied by a guest.



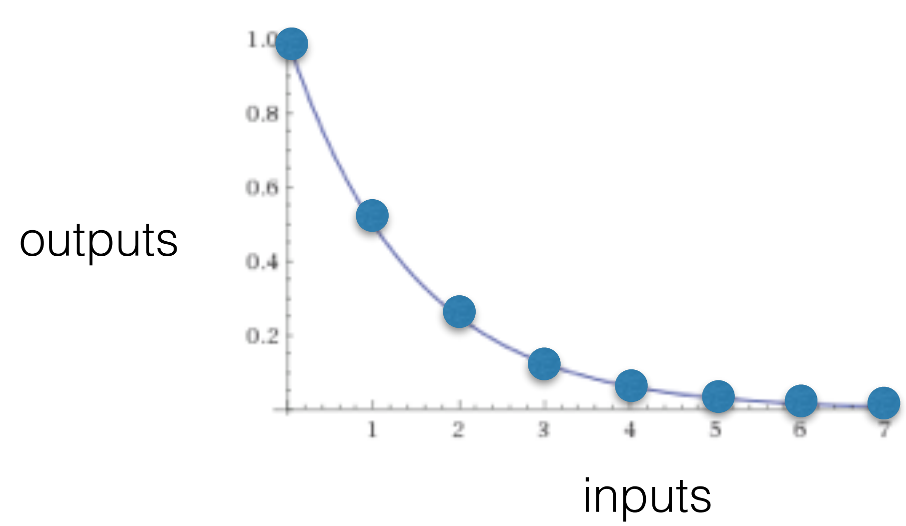
A new guest arrives. Can they get a room?
Turns out ... yes! You can just move every guest
from one room to the next and not run out of rooms.
Then, put the new guest in the first room.
Infinite sets can behave in counter-intuitive ways.

Sequences

An ***infinite sequence*** of numbers is actually just a function $f : \mathbf{N} \to B$ which maps the natural numbers {0, 1, 2, 3, ... } to some set B.

Here is a sequence of rational numbers: $f(n) = 1/2^n$
Its first few elements are 1, 1/2, 1/4, 1/8, 1/16, ...
As n (input) gets bigger, the output gets closer to 0.

$f : \mathbf{N} \to \mathbf{Q}$



Countably Infinite

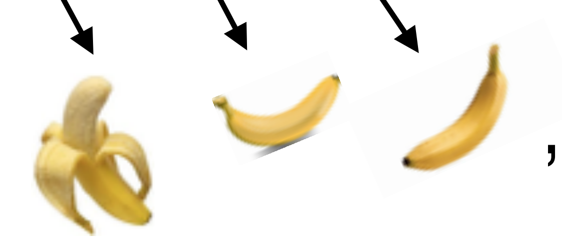
Mapping the natural numbers ***onto*** a set is like ***counting*** the elements in the set.

If we can count the elements in an infinite set B, we call it ***countably infinite***.

We just have to find a sequence $f: \mathbf{N} \rightarrow B$ that is ***onto***, meaning every member of B is found in the sequence.

Imagine a countably infinite set of bananas:

$\{1, \quad 2, \quad 3, \ldots\}$ domain $\mathbf{N}$

{  , ...} codomain B

Counting the Integers

It turns out the set **Z** (integers) is ***countably infinite***!

If you just tried to count all the positive integers first, you'd never get around to counting the negative ones.

But you can alternate positive/negative, like so:

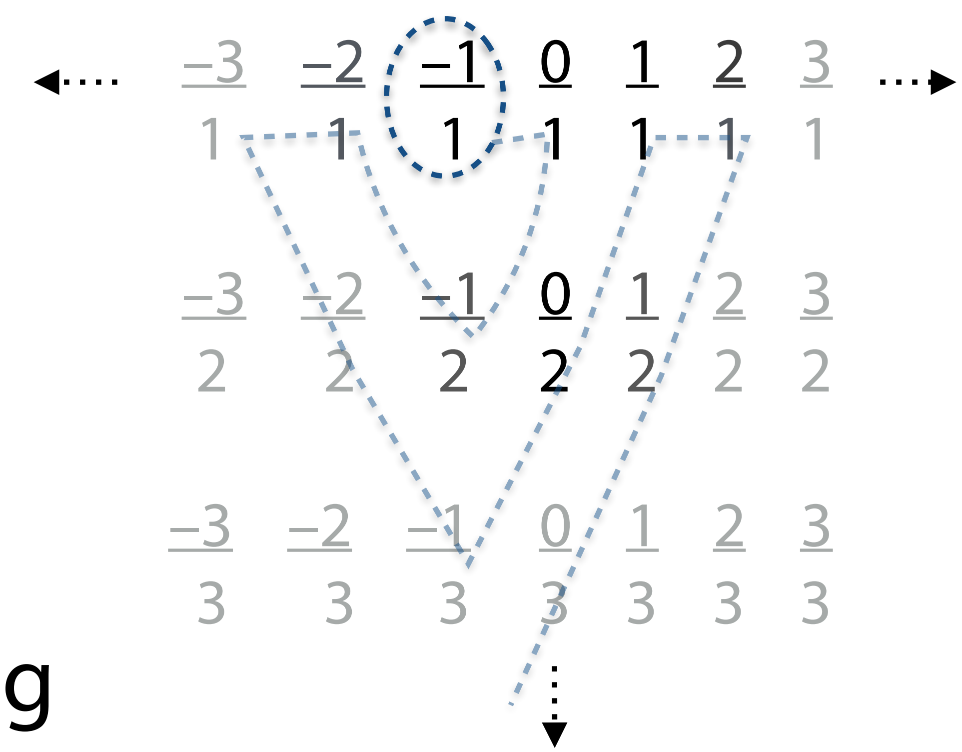
$$\begin{array}{ccccccl} \{0, & 1, & 2, & 3, & 4, & \ldots\} & \text{domain } \mathbf{N} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ \{0, & 1, & -1, & 2, & -2, & \ldots\} & \text{codomain } \mathbf{Z} \end{array}$$

In this way, the sequence maps **N** onto all of **Z**.
So $|\mathbf{N}| \geq |\mathbf{Z}|$. Since $\mathbf{N} \subseteq \mathbf{Z}$, it turns out that $|\mathbf{N}| = |\mathbf{Z}|$!!

Counting the Rationals

The set **Q** (rational numbers) is countably infinite, too.

This infinite grid contains all of the rational numbers, even though some of them are listed more than once (e.g. 1/2 = 2/4).



We construct a sequence that counts all rational numbers at least once, this time showing that |**N**| = |**Q**| !!

Union Countable Sets

If sets *A* and *B* are countably infinite, then their union is also countably infinite. Simply count like this:

$$a_1, b_1, a_2, b_2, a_3, \mathrm{b}_3, \ldots$$

This is true for the union of any number of countably infinite sets. In fact, a countable union of countable sets is still countable. You can arrange them like you did with **Q**, and count them that way.

So, are all infinities countable?

Reals are Uncountable!

We use a symbol to denote the cardinality of countably infinite sets: $|\mathbf{N}| = |\mathbf{Z}| = |\mathbf{Q}| = \aleph_0$

And it turns out that $|\mathbf{R}| > \aleph_0$

We can't even list all the points in the interval $0 \leq x \leq 1$.

Given any such sequence of real numbers, we can easily construct a number $0 \leq x \leq 1$ which is not in the sequence, by making it different by one digit from each number there:

```
0 → 0.012317...
1 → 0.527382...
2 → 0.832743...
3 → 0.419351...
...
x = 0.1334...
```

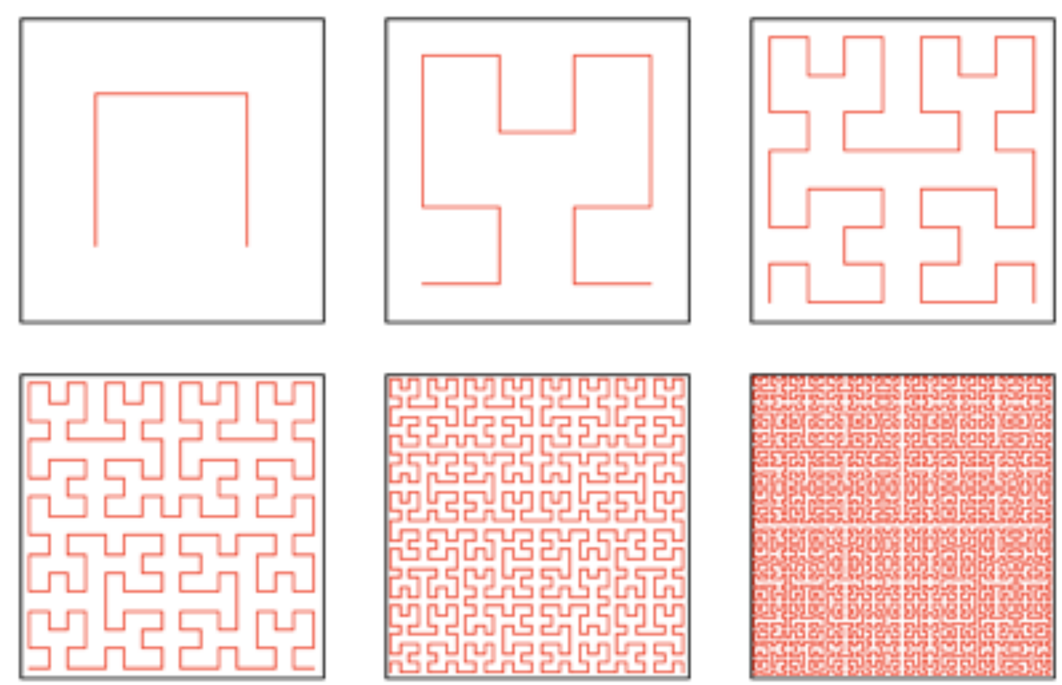
Complex Numbers?

We use a symbol to denote the cardinality of $|\mathbf{R}| = \mathfrak{c}$

That fancy *c* stands for "continuum".

So what is the cardinality of **C**, the set of all complex numbers? It turns out that $|\mathbf{C}| = |\mathbf{R}|$!

There are functions that can map a one-dimensional line onto a two dimensional plane in an invertible way. They are called space-filling curves.



Removing Elements

What if we removed 999 elements from an infinite set?

It would be still be infinite. Otherwise, if the resulting set had n elements, the infinite set would have $n + 999$ elements, which is a **contradiction**.

What if we took all the real numbers **R** and removed all the the rational numbers **Q**, leaving only irrationals?

The set of irrational numbers is uncountable! Otherwise, if it had been countable, then **R** = rational + irrational together would be countable, which is a **contradiction**.

advanced

Power Set

Given a set A, the collection of all its subsets is called the ***power set of A*** $2^A = \{ B : B \subseteq A \}$

For example the power set of {1, 2, 5} has 8 elements:
{ {}, {1}, {2}, {5}, {1,2}, {1,5}, {2,5}, {1,2,5} }

The power set of A has a far bigger size than A itself.
When A is a finite set, i.e. $|A| = n$, then $|2^A| = 2^n$

Each subset of A consists of some members of A.
Each $a \in A$ is either in a given subset of A or not: 2 possibilities per a.
Thus, if A has n elements, that's a total of 2^n different subsets.

advanced

Infinite Power Sets

What about an infinite set A? We can show that, in fact, its power set always has bigger cardinality than A!

$$|A| < |2^A|$$

Suppose you *could* actually find f mapping A ***onto*** 2^A.
So, f maps every member of a to a subset of A.
Well, then consider the set $B = \{ a \in A : a \notin f(a) \}$.
This is the set of all members of A which are *not* members of the corresponding subset $f(a) \subseteq A$.
Since every member of 2^A is some $f(a)$, so $B = f(b)$ for some b.
Now, if $b \in B$ then $b \notin f(b) = B$, which is a **contradiction**.
But if $b \notin B$ then $b \in f(b) = B$, which is also a **contradiction**.
This means no such function f could actually exist.

advanced

No Biggest Infinity

That this means is that there is no biggest infinite set. Given <u>any</u> set *A*, its power set would have an even bigger cardinality.

$$\aleph_0 < \aleph_1 < \aleph_2 < \ldots$$

$$\beth_0,\ \beth_1,\ \beth_2,\ \beth_3,\ \ldots$$

Concept: Infinity

In this topic, we explored the concept of infinity through considering functions on ***infinite sets***. We took concepts such as the ***size of a set***, which are easy and familiar when applied to finite sets, and worked out what it would mean for infinite sets. The resulting generalized concept is called ***cardinality***.

We also explored useful ***properties of functions*** that enable us to map one set onto another. Interestingly, it turned out that some infinite sets are the same size as their proper subsets. The sets **Z** and **Q** are the same size as the set of all natural numbers, even though they contain this set. You can count them all and never run out of natural numbers. There is a whole class of ***countably infinite*** sets, and their cardinality is denoted $\aleph_0$.

However, we then showed that **R** and **C** cannot be counted, because their cardinality is bigger. It is denoted $\mathfrak{c}$ for "continuum". But this is not the biggest cardinality, either. Given any set, we can construct its ***power set*** which always has an even larger cardinality. Thus, there is no biggest infinity in mathematics!